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Math 132: Differential Topology

§ Differential forms

Recall that a vector field v on M is a smoothly varying field of tangent vectors $v_x \in T_x M$.

<u>Def</u>
Likewise, a <u>1-form</u> α on M is a smoothly varying field of <u>cotangent vectors</u> $\alpha_x \in T_x^* M := (T_x M)^*$.

Note, if v is a vector field and α is a 1-form,

then $\alpha(v) : M \longrightarrow \mathbb{R}$ is a smooth function.
$x \longmapsto \alpha_x(v_x)$

<u>Ex</u> If $f: M \to \mathbb{R}$ is a smooth function, $df_x : T_x M \to \mathbb{R}$ is a linear map at each x, so df is a 1-form on M.

For any vector field v on M, $df(v) =: vf$ is a smooth function on M given by the directional derivative of f along v_x at each $x \in M$.

<u>Def</u> A <u>p-form</u> ω on M is a smoothly varying field of alternating p-tensors $\omega_x \in \Lambda^p(T_x^* M)$.

(A 0-form is just a smooth function.)

Ex In $\mathbb{R}^m$, we have coordinate functions $x_1, \cdots, x_m$ which yield 1-forms $dx_1, \cdots, dx_m$.

For each strictly increasing sequence $I = (i_1, \cdots, i_p)$, $1 \leq i_1 < \cdots < i_p \leq m$,

$$dx_I := dx_{i_1} \wedge \cdots \wedge dx_{i_p} \text{ is a } p\text{-form on } \mathbb{R}^m.$$

In fact, every p-form on an open subset $U \subset \mathbb{R}^m$ can be uniquely expressed as a sum $\sum_I f_I \, dx_I$ over increasing index sequences I, the f_I being functions on U.

Rmk Some properties of differential forms:

- (linearity) If ω_1 and ω_2 are p-forms, then their pointwise sum $\omega_1 + \omega_2$ is also a p-form
- (wedge product) If ω is a p-form and θ is a q-form, then $\omega \wedge \theta = (-1)^{pq} \theta \wedge \omega$ is a $(p+q)$-form.
- (pull back)* If $f: M \to N$ is a smooth map and ω is a p-form on N, then the pullback $f^*\omega$ of ω by f, defined by $(f^*\omega)_x = (df_x)^* \omega_{f(x)}$ is a p-form on M.

$$\left(\begin{tikzcd} \Lambda^p(T_xM) \arrow[r, "df_x"] \arrow[rr, bend right, "(f^*\omega)_x"'] & \Lambda^p(T_{f(x)}N) \arrow[r, "\omega_{f(x)}"] & \mathbb{R} \end{tikzcd} \right)$$

Pullback satisfies:

$$\begin{cases} f^*(\omega_1+\omega_2) = f^*\omega_1 + f^*\omega_2, \\ f^*(\omega\wedge\theta) = f^*\omega \wedge f^*\theta, \\ (f\circ g)^*\omega = g^* f^*\omega. \end{cases}$$

Ex Let's see explicitly what pullback f^* does on Euclidean spaces.

Let $f: V \to U$ be a smooth map between open subsets $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ with coordinate functions $x_1, \cdots, x_m$ on $\mathbb{R}^m$ and $y_1, \cdots, y_n$ on $\mathbb{R}^n$.

Then,
$$f^* dx_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial y_j} dy_j = df_i \leftarrow f_i = x_i \circ f$$

For an arbitrary p-form $\omega = \sum_I a_I \, dx_I$ on U,

$$f^*\omega = \sum_I (f^* a_I)(f^* dx_I) = \sum_I (a_I \circ f)\, df_I$$

where $df_I = df_{i_1} \wedge \cdots \wedge df_{i_p}$ for $I = (i_1, \cdots, i_p)$.

Ex In the special case when $f: V \to U$ is a diffeomorphism of two open subsets of $\mathbb{R}^m$,

$$\begin{aligned} f^*(\underbrace{dx_1 \wedge \cdots \wedge dx_m}_{\text{"volume form"}}) &= df_1 \wedge \cdots \wedge df_m \\ &= \det(df) \cdot dy_1 \wedge \cdots \wedge dy_m. \end{aligned}$$

We can use the pullback to define precisely what we mean by "smoothness" for a form ω:

- We say a form ω on an open set $U \subset \mathbb{R}^m$ is <u>smooth</u> if each coefficient function a_I in its expansion $\omega = \sum_I a_I dx_I$ is smooth.

  Note, if ω is smooth and $f: V \to U$ is a smooth map, then $f^*\omega$ is smooth as well.

- More generally, a form ω on M is <u>smooth</u> if, for every local parametrization $\varphi: U \to M$, $\varphi^*\omega$ is smooth on U.